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# Discrete Analysis for Plate Bending Problems by Using Hybrid-type Penalty Method

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SUMMARY

In present paper, we have given investigation of the plate bending problem by numerical treatment using hybrid-type penalty method (HPM) based on discontinuous Galerkin method. The HPM assume linear and nonlinear displacement field with rigid displacement, rigid rotation, strain and its gradient in each subdomain and introduce subsidiary condition about the continuity of displacement into the framework of the variational expression with Lagrange multipliers. For this purpose, we accept the Kirchhoff theory, which takes no into account the transversal shear deformation. In first step of the work, we give the equilibrium equations for deformable body in 3D case and as boundary conditions we give geometrical (for displacement field) and kinetic (for surface force) boundary conditions. Secondary we apply Kirchhoff theory to make the displacement field for plate bending problem into the 3D case. For this purpose, we use quadratic form, which includes rigid, linear, and nonlinear parts of displacements. It can define the parameters used in this displacement field as each subdomain independently. We introduce penalty function, which presents strong spring connecting each subdomain. Then we obtain stiffness matrix as every contact surface of each subdomain. The discretization equation of this model becomes a simulteniuos linear equation. The coefficient matrix consists of stiffness in the sub-domain and subsidiary condition on the intersection boundary for the adjacent sub-domain. In this model, it can express the discontinuous phenomenon of hinge etc. without changing degree of freedom. Finally, we compute simple problems to check the accuracy of the elastic solution.

KEY WORDS: plate bending, penalty method, hybrid-type virtual work, discontinuous Galerkin method

## 1. INTRODUCTION

In this work for obtaining numerical results for plate bending problem, we have used hybrid-type penalty method (HPM), which applied the concept of the penalty method [1] to the principle of hybrid type virtual work [2]. The HPM based on discontinuous Galerkin (dG) method [3]. HPM applies the concept of the spring of RBSM [4] (Rigid Bodies-Spring Model) in Lagrange multiplier and assume independent displacement field to each subdomain. Because compatibility requirements of the intersection boundary on adjacent sub-domain are secured by using the penalty method, the displacement field can be assumed regardless of the shape of sub-domain [5]. However, it cannot obtain high accuracy solutions when it uses shape other than the triangle at the linear displacement field. For the reason, use of the mesh division of arbitrary shape is difficult. To solve such a problem, it proposes the method of applying the second-order displacement field where added the gradient of the strain to linear displacement field of HPM [6].

Usually for finite element method, it is requiring $C^1$ continuity. However many plate elements and results obtained are not completely satisfactory. So alternatively using $C^0$ elements and impose the continuity of slope weakly. This method called discontinuous Galerkin method [7].

In the following paper, we proposed new element model for plate bending problems by using HPM based on the dG method. In the first part of the paper, we have given brief formulation of proposed method, and in second part, we have given some numerical results.



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## 2. GOVERNING EQUATION AND HYBRID TYPE VIRTUAL WORK

### *2.1 Governing Equation*

Let $\Omega \subset \mathbb{R}^{\mathrm{n_{dim}}}$, with $(1 \leq \mathrm{n_{dim}} \leq 3)$, be the reference configuration of a continuum body with smooth boundary $\Gamma := \partial\Omega$ and closure $\overline{\Omega} := \Omega \cup \partial\Omega$. Here $\mathbb{R}^{\mathrm{n_{dim}}}$ is the $\mathrm{n_{dim}}$ dimensional Euclidean space.

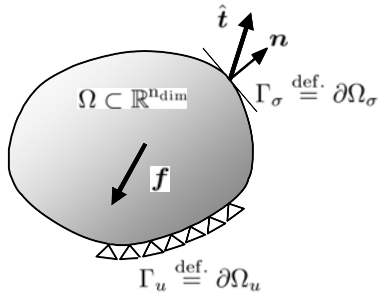


*Fig. 1. Reference configuration $\Omega$ and smooth boundary $\partial\Omega$*

The local form of the equilibrium equation for a deformable body is as follows:

$$\operatorname{div} \boldsymbol{\sigma} + \boldsymbol{f} = 0 \quad \text{in} \quad \Omega \tag{1}$$

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}^t \quad \text{in} \quad \Omega \tag{2}$$

where $\boldsymbol{f} : \Omega \to \mathbb{R}^{\mathrm{n_{dim}}}$ is the body force per unit volume, $\boldsymbol{\sigma} : \overline{\Omega} \to \mathbb{S}$ is the Cauchy stress tensor respectively. Here $\mathbb{S} = \mathbb{R}^{(\mathrm{n_{dim}}+1)\cdot \mathrm{n_{dim}}/2}$ is the vector space of symmetric rank-two tensor and $\boldsymbol{e}_i$ is the standard base vector of $\mathbb{R}^{\mathrm{n_{dim}}}$, so that the stress tensor becomes $\boldsymbol{\sigma} = \sigma_{ij} \boldsymbol{e}_i \otimes \boldsymbol{e}_j$, where $\otimes$ denotes a tensor product. $\boldsymbol{u} : \overline{\Omega} \to \mathbb{R}^{\mathrm{n_{dim}}}$ is a displacement field of particles with reference position $\boldsymbol{x} \in \Omega$. We write $\boldsymbol{u}(\boldsymbol{x})$ and denote the infinitesimal strain tensor by

$$\boldsymbol{\varepsilon} = \boldsymbol{\nabla}^s \boldsymbol{u} \overset{\text{def.}}{=} \frac{1}{2} \left[ \boldsymbol{\nabla} \boldsymbol{u} + (\boldsymbol{\nabla} \boldsymbol{u})^t \right] \tag{3}$$

where $\boldsymbol{\nabla} := (\partial / \partial x_i) \boldsymbol{e}_i$ is the differential vector operator, $\boldsymbol{\nabla}^s$ shows the symmetry part of $\boldsymbol{\nabla}$.
In what follows, we assume that the boundary $\Gamma = \Gamma_u \cup \Gamma_\sigma$.

$$\overline{\Gamma} = \overline{\Gamma_u \cup \Gamma_\sigma} \;\; , \;\; \Gamma_u \cap \Gamma_\sigma = \emptyset \tag{4}$$

Here $\Gamma_u := \partial_u \Omega \subset \partial\Omega$ where displacement are prescribed as

$$\boldsymbol{u}|_{\Gamma_u} = \hat{\boldsymbol{u}} \;\; \text{(given)} \tag{5}$$

Where as $\Gamma_\sigma := \partial_\sigma \Omega \subset \partial\Omega$ where tractions $\boldsymbol{t} := \boldsymbol{\sigma} \boldsymbol{n}$ are prescribed as

$$\boldsymbol{\sigma}|_{\Gamma_\sigma} \hat{\boldsymbol{n}} = \hat{\boldsymbol{t}} \;\; \text{(given)} \tag{6}$$

Here $\hat{\boldsymbol{n}}$ is the field normal to the boundary $\Gamma_\sigma$. The constitutive equation to the elastic body is provided as follows by the use of the elasticity tensor $\boldsymbol{C}$.

$$\boldsymbol{\sigma} = \boldsymbol{C} : \boldsymbol{\varepsilon} \tag{7}$$

### *2.2 Virtual Work Equation (Weak Forms)*

Let denote by $\mathbb{U}$ the space of admissible displacement field, defined as

$$\mathbb{U} \overset{\text{def.}}{=} \{ \boldsymbol{u} : \Omega \to \mathbb{R}^{\mathrm{n_{dim}}} \mid \boldsymbol{u}|_{\Gamma_u} = \hat{\boldsymbol{u}} \} \tag{8}$$

Moreover, let denote by $\mathbb{V}$ the space of admissible virtual displacement field, defined as

$$\mathbb{V} \overset{\text{def.}}{=} \{ \delta\boldsymbol{u} : \Omega \to \mathbb{R}^{\mathrm{n_{dim}}} \mid \delta\boldsymbol{u}|_{\Gamma_u} = \boldsymbol{0} \} \tag{9}$$

We can now use Equation (1) and integrate volume of the body to give a weak statement of the static equilibrium of the body as

$$\delta W := \int_{\Omega} (\operatorname{div} \boldsymbol{\sigma} + \boldsymbol{f}) \cdot \delta \boldsymbol{u} \, dV = 0 \qquad {}^{\forall}\delta \boldsymbol{u} \in \mathbb{V} \tag{10}$$

It can derive a more common and useful expression to give the divergence of the vector $\boldsymbol{\sigma}\delta\boldsymbol{u}$ as

$$\operatorname{div}(\boldsymbol{\sigma}\delta\boldsymbol{u}) = (\operatorname{div}\boldsymbol{\sigma}) \cdot \delta\boldsymbol{u} + \boldsymbol{\sigma} : \operatorname{grad} \delta\boldsymbol{u} \tag{11}$$

Using this equation together with the Gauss theorem enable Equation (10) to be rewritten as

$$\int_{\Omega} \boldsymbol{\sigma} : \operatorname{grad}\delta\boldsymbol{u} \, dV - \int_{\Omega} \boldsymbol{f} \cdot \delta\boldsymbol{u} dV - \int_{\Gamma_{\sigma}} \hat{\boldsymbol{t}} \cdot \delta\boldsymbol{u} \, dS = 0 \qquad {}^{\forall}\delta\boldsymbol{u} \in \mathbb{V} \tag{12}$$

This equation is virtual work equation. If $\boldsymbol{u}$ is the weighing function, this is a weak forms. It is $\mathbb{U} \subset \mathbb{H}^1(\Omega)$ and $\mathbb{V} \subset \mathbb{H}^1(\Omega)$ where denotes the Sobolev space $\mathbb{H}^1(\Omega)$ of function possessing space integrable derivatives.

*2.3 Hybrid Type Virtual Work Equation*

Let $\Omega$ consist of M sub-domains $\Omega^{(e)} \subset \Omega$ with the closed boundary $\Gamma^{(e)} := \partial\Omega^{(e)}$ as shown in Figure 2.

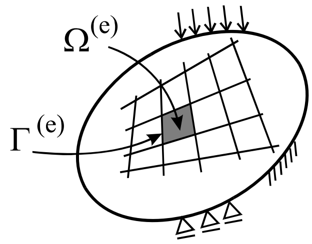


*Fig. 2. Sub-domain* $\Omega^{(e)}$

That is

$$\Omega = \bigcup_{e=1}^{M} \Omega^{(e)} \qquad \text{here} \quad \Omega^{(r)} \cap \Omega^{(q)} = 0 \quad (r \neq q) \tag{13}$$

In what follows, we assume that the closure $\overline{\Omega}^{(e)} := \Omega^{(e)} \cup \partial\Omega^{(e)}$.
We denoted by $\Gamma_{<ab>}$ the common boundary for two subdomain $\Omega^{(a)}$ and $\Omega^{(b)}$ adjoined as shown in Figure 3, and which is defining as

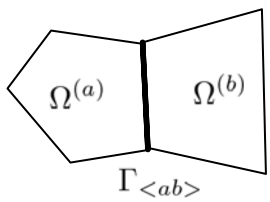


*Fig. 3. Common boundary* $\Gamma_{<ab>}$ *of sub-domain* $\Omega^{(a)}$ *and* $\Omega^{(b)}$

$$\Gamma_{<ab>} \stackrel{\text{def.}}{=} \Gamma^{(a)} \cap \Gamma^{(b)} \tag{14}$$

The relation for $\tilde{\boldsymbol{u}}^{(a)}$ and $\tilde{\boldsymbol{u}}^{(b)}$ are following:

$$\tilde{\boldsymbol{u}}^{(a)} = \tilde{\boldsymbol{u}}^{(b)} \qquad \text{on} \quad \Gamma_{<ab>} \tag{15}$$

They are the displacements on $\Gamma_{<ab>}$ that is intersection boundary in $\Omega^{(a)}$ and $\Omega^{(b)}$ sub-domain. It introduces this subsidiary condition into the framework of the variational equation (12) with Lagrange multipliers $\boldsymbol{\lambda}$ as follows:

$$H_{ab} \stackrel{\text{def.}}{=} \delta \int_{\Gamma_{<ab>}} \boldsymbol{\lambda} \cdot (\tilde{\boldsymbol{u}}^{(a)} - \tilde{\boldsymbol{u}}^{(b)}) \, dS \tag{16}$$

where $\delta(\bullet)$ shows the variation of $(\bullet)$. Physical meaning of the Lagrange multiplier $\boldsymbol{\lambda}$ is equal to the surface force on the intersection boundary $\Gamma_{<ab>}$.

$$\boldsymbol{\lambda} = \boldsymbol{t}^{(a)}(\widetilde{\boldsymbol{u}}^{(a)}) = -\boldsymbol{t}^{(b)}(\widetilde{\boldsymbol{u}}^{(b)}) \tag{17}$$

where $\boldsymbol{t}^{(a)}$ and $\boldsymbol{t}^{(b)}$ are the surface force on the intersection boundary $\Gamma_{<ab>}$ in sub-domain $\Omega^{(a)}$ and $\Omega^{(b)}$. As described above, the hybrid type virtual work equation is as follows about *M* subdomain and *N* intersection boundary:

$$\sum_{e=1}^{M}\left(\int_{\Omega^{(e)}} \boldsymbol{\sigma} : \mathrm{grad}(\delta \boldsymbol{u}) dV - \int_{\Omega^{(e)}} \boldsymbol{f} \cdot \delta \boldsymbol{u}\, dV\right) - \sum_{s=1}^{N}\left(\delta \int_{\Gamma_{<s>}} \boldsymbol{\lambda} \cdot (\widetilde{\boldsymbol{u}}^{(a)} - \widetilde{\boldsymbol{u}}^{(b)})\, dS\right) - \int_{\Gamma_\sigma} \hat{\boldsymbol{t}} \cdot \delta \boldsymbol{u}\, dS = 0 \quad \forall \delta \boldsymbol{u} \in \mathbb{V} \tag{18}$$

## 3. INDEPENDENT DISPLACEMENT FIELD AND RELATIVE DISPLACEMENT

### *3.1 3D Displacement Field*

In the following work, it considers a three-dimensional displaced field $\boldsymbol{u} \in \mathbb{U}$ with $n_{\mathrm{dim}} = 3$. It carries out Taylor's expansion of this displacement $\boldsymbol{u}(\boldsymbol{x})$ about the any point $\boldsymbol{x}_p = (x_p, y_p, z_p) \in \Omega^{(e)}$ in a domain $\Omega^{(e)}$. Consequently, the second-order displaced field in the arbitrary sub-domains $\Omega^{(e)}$ by matrix form will be:

$$\boldsymbol{u}^{(e)} = \boldsymbol{N}_d^{(e)} \boldsymbol{d}^{(e)} + \boldsymbol{N}_\varepsilon^{(e)} \boldsymbol{\varepsilon}^{(e)} + \boldsymbol{N}_{g_x}^{(e)} \boldsymbol{\varepsilon}_{,x}^{(e)} + \boldsymbol{N}_{g_y}^{(e)} \boldsymbol{\varepsilon}_{,y}^{(e)} + \boldsymbol{N}_{g_z}^{(e)} \boldsymbol{\varepsilon}_{,z}^{(e)} \tag{19}$$

where

$$\boldsymbol{N}_d{}^{(e)} = \begin{bmatrix} 1 & 0 & 0 & 0 & Z & -Y \\ 0 & 1 & 0 & -Z & 0 & X \\ 0 & 0 & 1 & Y & -X & 0 \end{bmatrix},\ \boldsymbol{N}_\varepsilon{}^{(e)} = \begin{bmatrix} X & 0 & 0 & \frac{Y}{2} & 0 & \frac{Z}{2} \\ 0 & Y & 0 & \frac{X}{2} & \frac{Z}{2} & 0 \\ 0 & 0 & Z & 0 & \frac{Y}{2} & \frac{X}{2} \end{bmatrix},\ \boldsymbol{N}_{g_x}{}^{(e)} = \begin{bmatrix} \frac{X^2}{2} & \frac{-Y^2}{2} & \frac{-Z^2}{2} & 0 & \frac{-YZ}{2} & 0 \\ 0 & XY & 0 & \frac{X^2}{2} & \frac{ZX}{2} & 0 \\ 0 & 0 & ZX & 0 & \frac{XY}{2} & \frac{X^2}{2} \end{bmatrix},$$

$$\boldsymbol{N}_{g_y}{}^{(e)} = \begin{bmatrix} XY & 0 & 0 & \frac{Y^2}{2} & 0 & \frac{YZ}{2} \\ \frac{-X^2}{2} & \frac{Y^2}{2} & \frac{-Z^2}{2} & 0 & 0 & \frac{-ZX}{2} \\ 0 & 0 & YZ & 0 & \frac{Y^2}{2} & \frac{XY}{2} \end{bmatrix},\ \boldsymbol{N}_{g_z}{}^{(e)} = \begin{bmatrix} ZX & 0 & 0 & \frac{YZ}{2} & 0 & \frac{Z^2}{2} \\ 0 & YZ & 0 & \frac{ZX}{2} & \frac{Z^2}{2} & 0 \\ \frac{-X^2}{2} & \frac{-Y^2}{2} & \frac{Z^2}{2} & \frac{-XY}{2} & 0 & 0 \end{bmatrix},$$

$$\boldsymbol{d} = [u_p, v_p, w_p, \theta_x, \theta_y, \theta_z]^t, \qquad \boldsymbol{\varepsilon} = [\varepsilon_x, \varepsilon_y, \varepsilon_z, \gamma_{xy}, \gamma_{yz}, \gamma_{zx}]^t,$$

$$\boldsymbol{\varepsilon}_{,x} = [\varepsilon_{x,x}, \varepsilon_{y,x}, \varepsilon_{z,x}, \gamma_{xy,x}, \gamma_{yz,x}, \gamma_{zx,x}]^t, \quad \boldsymbol{\varepsilon}_{,y} = [\varepsilon_{x,y}, \varepsilon_{y,y}, \varepsilon_{z,y}, \gamma_{xy,y}, \gamma_{yz,y}, \gamma_{zx,y}]^t,$$

$$\boldsymbol{\varepsilon}_{,z} = [\varepsilon_{x,z}, \varepsilon_{y,z}, \varepsilon_{z,z}, \gamma_{xy,z}, \gamma_{yz,z}, \gamma_{zx,z}]^t,$$

$$X = x - x_p, Y = y - y_p, Z = z - z_p$$

### *3.2 Displacement for Thin Plate*

The thin plate is defining as follows:

$$\Omega = \{(x, y, z) \in R^3 | z \in [-t/2, t/2], (x, y) \in A \in R^2\} \tag{20}$$

where $t$ is plate thickness and $A$ is plate area.

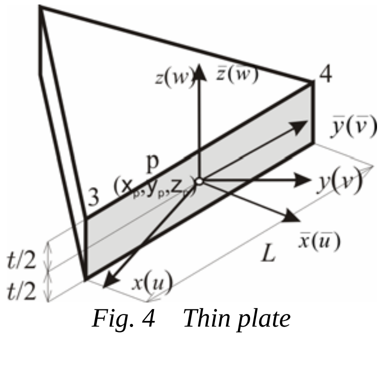


*Fig. 4 Thin plate*

Since for thin plate we have following Kirchhoff-Love's assumptions:

$$\varepsilon_z = \gamma_{yz} = \gamma_{zx} = 0 \tag{21}$$

we will obtain deflection for thin plate as follows:

$$w = w_p + Y\theta_x^p - X\theta_y^p - \frac{1}{2}X^2\varepsilon_{x,z}^p - \frac{1}{2}Y^2\varepsilon_{y,z}^p - \frac{1}{2}XY\gamma_{xy,z}^p \tag{22}$$

Moreover, displacement at arbitrary point will be:

$$\boldsymbol{u}^{(e)} = \boldsymbol{Z}_M \boldsymbol{N}_{Md}^{(e)} \boldsymbol{d}_M^{(e)} + \boldsymbol{Z}_M \boldsymbol{N}_{Mg}^{(e)} \boldsymbol{\varepsilon}_M^{(e)} \tag{23}$$

where

$$\boldsymbol{Z}_M = \begin{bmatrix} -z & 0 & 0 \\ 0 & -z & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \boldsymbol{N}_{Md}^{(e)} = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & Y & -X \end{bmatrix}, \quad \boldsymbol{N}_{Mg}^{(e)} = \begin{bmatrix} -X & 0 & -\frac{Y}{2} \\ 0 & -Y & -\frac{X}{2} \\ -\frac{X^2}{2} & -\frac{Y^2}{2} & -\frac{XY}{2} \end{bmatrix},$$

$$\boldsymbol{d}_M^{(e)} = \begin{Bmatrix} w_p \\ \theta_x^p \\ \theta_y^p \end{Bmatrix}, \quad \boldsymbol{\varepsilon}_M^{(e)} = \begin{Bmatrix} \varepsilon_{x,z}^p \\ \varepsilon_{y,z}^p \\ \varepsilon_{xy,z}^p \end{Bmatrix}, \quad \boldsymbol{u}^{(e)} = \begin{Bmatrix} u^{(e)} \\ v^{(e)} \\ w^{(e)} \end{Bmatrix}$$

It can write Equation (23) as follows by matrix form:

$$\boldsymbol{u}^{(e)} = \boldsymbol{Z}_M \boldsymbol{N}^{(e)} \boldsymbol{U}^{(e)} \tag{24}$$

where

$$\boldsymbol{N}^{(e)} = \lfloor \boldsymbol{N}_{Md}^{(e)} \boldsymbol{N}_{Mg}^{(e)} \rfloor \quad , \quad \boldsymbol{U}^{(e)} = \lfloor \boldsymbol{d}_M^{(e)} \boldsymbol{\varepsilon}_M^{(e)} \rfloor^t$$

*3.3 Relative Displacement*

Now we have to transform into local coordinate system from global coordinate system. This relation is as follows by matrix form:

$$\tilde{\boldsymbol{u}}^{(e)} = \boldsymbol{R}^{(e)} \boldsymbol{u}^{(e)} \tag{25}$$

where $\boldsymbol{R}^{(e)}$ is:

$$\boldsymbol{R}^{(e)} = \begin{bmatrix} l & m & 0 \\ -m & l & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

where

$$l = \frac{y_{43}}{L}, m = -\frac{x_{43}}{L}, L = \sqrt{{x_{43}}^2 + {y_{43}}^2}, \quad x_{ij} = x_i - x_j$$

The relative displacement on the intersection boundary will be:

$$\boldsymbol{\delta}_{<ab>} = \sum \boldsymbol{R}_{<ab>}^{(e)} \boldsymbol{U}_{<ab>}^{(e)} \tag{26}$$

and the matrix form for the relative displacement is:

$$\boldsymbol{\delta}_{<ab>} = \boldsymbol{Z}_M \boldsymbol{B}_{<ab>} \boldsymbol{U}_{<ab>} \tag{27}$$

where

$$\boldsymbol{B}_{<ab>}^{(e)} = \lfloor \boldsymbol{R}_{<ab>}^{(a)} \boldsymbol{N}^{(a)} \boldsymbol{R}_{<ab>}^{(b)} \boldsymbol{N}^{(b)} \rfloor \quad , \quad \boldsymbol{U}_{<ab>} = \lfloor \boldsymbol{U}^{(a)} \boldsymbol{U}^{(b)} \rfloor$$

## 4. DISCRETIZATION FOR HPM

*4.1 Lagrange Multiplier and Penalty Function*

Physical meaning of the Lagrange multiplier $\boldsymbol{\lambda}$ is equal to the surface force on the intersection boundary. Generally, in a hybrid-type variational principle, it deals in this multiplier as an unknown parameter. Since it has the meaning that Lagrange multiplier $\boldsymbol{\lambda}$ is the surface force on the boundary $\Gamma_{<ab>}$ in sub-domain $\Omega^{(a)}$ and $\Omega^{(b)}$, the surface force is defined as follows:

$$\boldsymbol{\lambda}_{<ab>} = \boldsymbol{k} \cdot \boldsymbol{\delta}_{<ab>} \tag{28}$$

Here $\boldsymbol{\delta}_{<ab>}$ shows relative displacement on the sub-domain boundary $\Gamma_{<ab>}$, and it is as follows in three dimensional problems (also plate bending problem):

$$\begin{Bmatrix} \lambda_{n<ab>} \\ \lambda_{sx<ab>} \\ \lambda_{sy<ab>} \end{Bmatrix} = \begin{bmatrix} k_n & 0 & 0 \\ 0 & k_{sx} & 0 \\ 0 & 0 & k_{sy} \end{bmatrix} \begin{Bmatrix} \delta_{n<ab>} \\ \delta_{sx<ab>} \\ \delta_{sy<ab>} \end{Bmatrix} \tag{29}$$

where

$$k_n = k_{sx} = k_{sy} = p$$

where *p* is a penalty function.

*4.2 Discretization for Subsidiary Condition*

We can write an Equation (16) as follows:

$$\boldsymbol{H}_{ab} = -\delta \int_{\Gamma_{<ab>}} \boldsymbol{\lambda}^t_{<ab>} (\boldsymbol{u}^{(a)}_{<ab>} - \boldsymbol{u}^{(b)}_{<ab>}) d\Gamma$$

$$= -\delta \int_{\Gamma_{<ab>}} \boldsymbol{\delta}^t_{<ab>} \boldsymbol{k}_{<ab>} \boldsymbol{\delta}_{<ab>} d\Gamma = -\delta \boldsymbol{U}^t_{<ab>} \int_{\Gamma_{<ab>}} \boldsymbol{B}^t_{<ab>} \boldsymbol{k}_{<ab>} \boldsymbol{B}_{<ab>} d\Gamma \boldsymbol{U}_{<ab>} \tag{30}$$

where

$$\boldsymbol{U}_{<ab>} = \boldsymbol{\mathcal{M}}_{<ab>} \boldsymbol{U}$$

here $\boldsymbol{\mathcal{M}}$ is a matrix which relates the total degree of freedom and the degree of freedom of each sub-domain. It is similar for virtual displacement:

$$\delta \boldsymbol{U}_{<ab>} = \boldsymbol{\mathcal{M}}_{<ab>} \delta \boldsymbol{U}$$

Then we obtain following equation:

$$\boldsymbol{H}_{ab} = -\delta \boldsymbol{U}^t \boldsymbol{K}_{<s>} \boldsymbol{U} \tag{31}$$

where

$$\boldsymbol{K}_{<s>} = \boldsymbol{\mathcal{M}}^t_{<s>} \int_{\Gamma_{xy<ab>}} \boldsymbol{B}^t_{<s>} \bar{\boldsymbol{k}}_{<s>} \boldsymbol{B}_{<s>} d\Gamma_{xy} \boldsymbol{\mathcal{M}}_{<s>}$$

where

$$\bar{\boldsymbol{k}}_{<s>} = \int_{-t/2}^{t/2} \boldsymbol{Z}^t_M \boldsymbol{k} \boldsymbol{Z}_M dZ = \begin{bmatrix} \frac{t^3}{12} k_n & 0 & 0 \\ 0 & \frac{t^3}{12} k_{sx} & 0 \\ 0 & 0 & t k_{sy} \end{bmatrix}$$

*4.3 Discretization Virtual Work Equation for Each Sub-Domain*

For thin plate theory a reduced form of the constitutive relations is obtained by making $\boldsymbol{\sigma}_z = 0$ and subsequently eliminating $\boldsymbol{\varepsilon}_z$.

Strains in thin plate with Kirchhoff-Love's assumption are:

$$\varepsilon_x = \frac{\partial u}{\partial x} = -z \frac{\partial^2 w}{\partial x^2}, \quad \varepsilon_y = \frac{\partial v}{\partial y} = -z \frac{\partial^2 w}{\partial y^2},$$

$$\gamma_{xy} = \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} = -2z \frac{\partial^2 w}{\partial x \partial y}, \gamma_{xz} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} = 0, \gamma_{yz} = \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} = 0 \tag{32}$$

After application of thin plate strains, we obtain $\bar{D}$ matrix for an elastic isotropic material:

$$\bar{\boldsymbol{D}} = \frac{E}{1 - \nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{(1-\nu)}{2} \end{bmatrix} \tag{33}$$

Next, it brings strain vector using displacement field.

$$\bar{\boldsymbol{\varepsilon}}^{(e)} = \boldsymbol{L}\boldsymbol{u}^{(e)} = \boldsymbol{Z}\boldsymbol{B}^{(e)}\boldsymbol{U}^{(e)} \tag{34}$$

where

$$\boldsymbol{B}^{(e)} = \boldsymbol{L}\boldsymbol{N}^{(e)}$$

We will have:

$$\int_{\Omega^{(e)}} [\boldsymbol{L}\delta\boldsymbol{u}]^t \boldsymbol{\sigma} d\Omega = \int_{\Omega^{(e)}} \delta\bar{\boldsymbol{\varepsilon}}^{(e)} \bar{\boldsymbol{D}}^{(e)} \bar{\boldsymbol{\varepsilon}}^{(e)} d\Omega = (\delta\boldsymbol{U}^{(e)})^t \int_{\Omega_{xy}^{(e)}} {\boldsymbol{B}^{(e)}}^t \boldsymbol{k}^{(e)} \boldsymbol{B}^{(e)} d\Omega_{xy} \boldsymbol{U}^{(e)} \tag{35}$$

We have:

$$\boldsymbol{U}^{(e)} = \boldsymbol{\mathcal{A}}^{(e)}\boldsymbol{U}, \qquad \delta\boldsymbol{U}^{(e)} = \boldsymbol{\mathcal{A}}^{(e)}\delta\boldsymbol{U}$$

As mentioned above, it obtains the following:

$$\int_{\Omega^{(e)}} [\boldsymbol{L}\delta\boldsymbol{u}]^t \boldsymbol{\sigma} d\Omega = \delta\boldsymbol{U}\boldsymbol{K}^{(e)}\boldsymbol{U} \tag{36}$$

here $\boldsymbol{\mathcal{A}}^{(e)}$ is a matrix which relates the total degree of freedom and the degree of freedom of each sub-domain.

$$\boldsymbol{K}^{(e)} = \int_{\Omega_{xy}^{(e)}} {\boldsymbol{B}^{(e)}}^t \boldsymbol{k}^{(e)} \boldsymbol{B}^{(e)} d\Omega_{xy}$$

here

$$\boldsymbol{k}^{(e)} = \int_{t/2}^{t/2} \boldsymbol{Z}^t \bar{\boldsymbol{D}}^{(e)} \boldsymbol{Z} dZ = \frac{t^3}{12}\frac{E}{1-\nu^2}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix} \tag{37}$$

The discretization of the body and surface force is as follows:

$$\int_{\Omega^{(e)}} \delta\boldsymbol{u}^t \boldsymbol{f} d\Omega + \int_{\Gamma^{(e)}} \delta\boldsymbol{u}^t \boldsymbol{T} d\Omega = \delta\boldsymbol{U}^t \boldsymbol{P}^{(e)} \tag{38}$$

where

$$\boldsymbol{P}^{(e)} = (\boldsymbol{\mathcal{A}}^{(e)})^t \int_{\Omega^{(e)}} (\boldsymbol{N}^{(e)})^t \boldsymbol{Z}_M \boldsymbol{f} d\Omega + \int_{\Gamma^{(e)}} (\boldsymbol{N}^{(e)})^t \boldsymbol{Z}_M \boldsymbol{T} d\Gamma$$

Finally, we obtain following discretized equation:

$$\delta\boldsymbol{U}^t(\sum \boldsymbol{K}^{(e)} + \sum \boldsymbol{K}_{<s>})\boldsymbol{U} - \delta\boldsymbol{U}^t(\sum \boldsymbol{P}^{(e)}) = 0 \tag{39}$$

Since $\delta\boldsymbol{U}$ is arbitrary, we can write Equation (39) as follows:

$$\boldsymbol{K}\boldsymbol{U} = \boldsymbol{P} \tag{40}$$

where

$$\boldsymbol{K} = \sum \boldsymbol{K}^{(e)} + \sum \boldsymbol{K}_{<s>} \quad , \quad \boldsymbol{P} = \sum \boldsymbol{P}^{(e)}$$

The discretization equation of this method becomes a simulteniuos linear equation shown in equation (40). Left coefficient matrix $\boldsymbol{K}$ consists of stiffness in the sub-domain and subsidiary condition on the intersection boundary for the adjacent sub-domain. It can express the discontinuous phenomenon of hinge etc., without changing degree of freedom by changing the value of $k$ of equation (28) to zero.

## 5. NUMERICAL EXAMPLE

As a numerical example, we present some simple problems.

At first, it gives cantilever beam with line-load. The beam has following material properties and geometrical properties:

Young's Modulus = 1x10$^6$ kN/m$^2$ ,
Poison's ratio = 0,
length=4m, width=1m,
thickness=0.1m,
line-load=1kN/m.

For this case, it obtains the deflection and the moments by exact treatment and by HPM. In the figure 5, it compares exact solution with numerical solution by HPM. Moreover, as shown in figure 5 the numerical solution for the moments exactly equal to the analytically solution. Also as shown in figure 6 numerical result of the deflection is high accuracy.

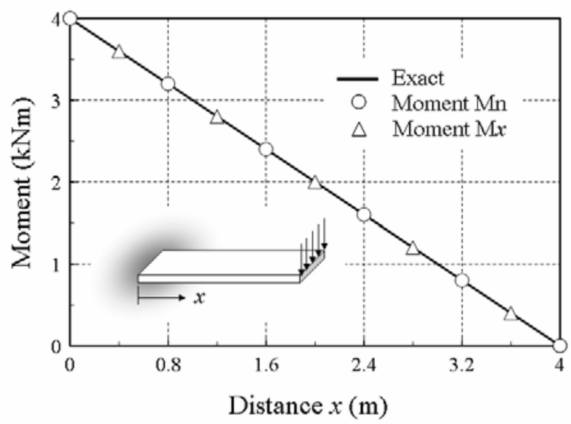


*Fig. 5 Moments*

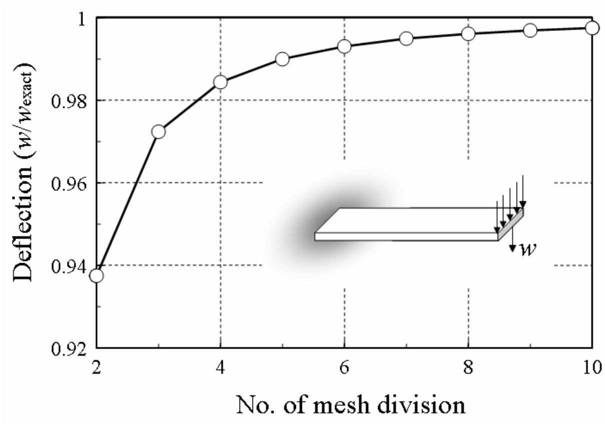


*Fig. 6 Ratio of deflections by exact and HPM*

It also compute both-end fixed beam with uniform distributed load. The beam has the same characteristics as in cantilever beam, only uniform-distributed-load=1kN/m$^2$. As for cantilever beam case, it has also given comparison between exact solution and HPM results. The results for this case have given by the figures 7 and 8. Here also for the moment analytical and numerical solutions are equal.

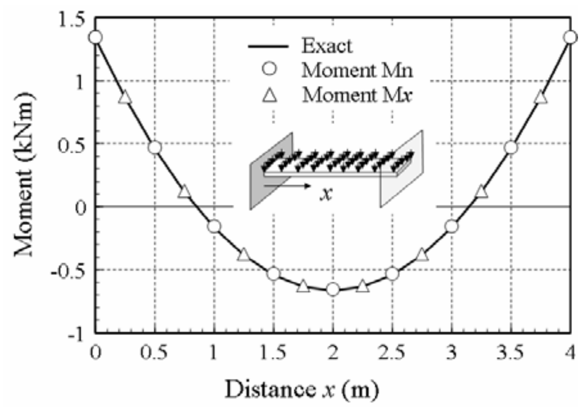


*Fig.7 Moments*

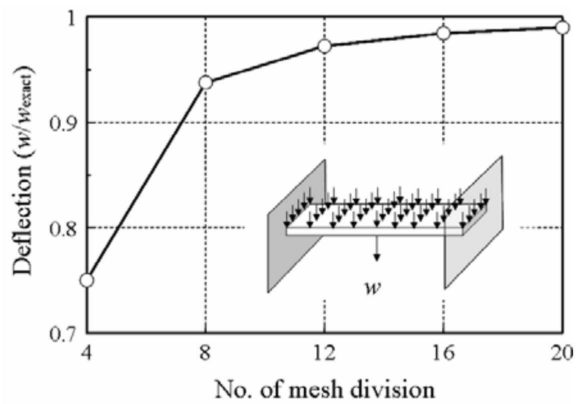


*Fig. 8 Ratio of deflections by exact and HPM*

The next example is simple supported plate with concentrated load. The plate material and the geometrical properties are following:

Young's Modulus = 1x10$^6$ kN/m$^2$
Poison's ratio = 0,
length=4m, width=4m and thickness=0.1m,
concentrated-load=4kN.

In figure 9 and 10, it show the results obtained for this case. Here we obtain high accuracy between exact and HPM results for moment calculations, but not exactly the same solutions.

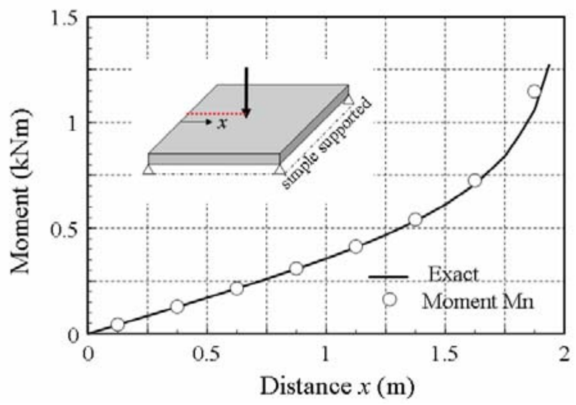

*Fig.9 Moments*

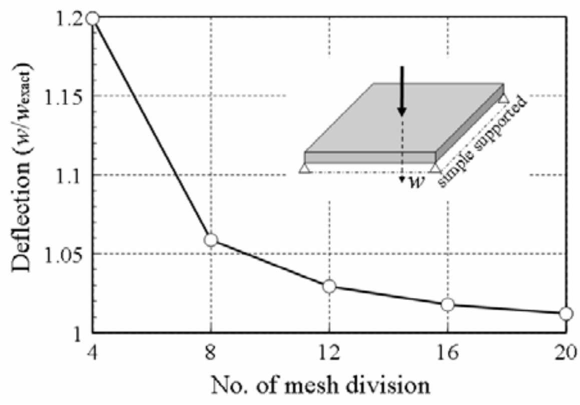

*Fig. 10 Ratio of deflections by exact and HPM*

The last example is again simple-supported plate, but with uniform distributed force, which is equal 1kN/m$^2$. Material and geometrical characteristics are the similar with previews example. It gives the results for this case in figures 11 and 12.

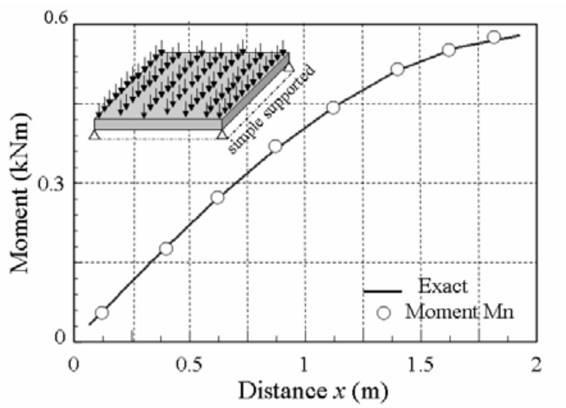

*Fig.11 Moments*

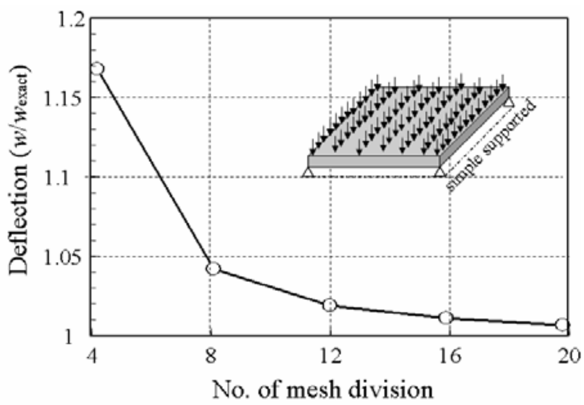

*Fig. 12 Ratio of deflections by exact and HPM*

These all calculations have done when Poisson Ration is equal zero. It gives the results for different cases of Poisson Ratio below.

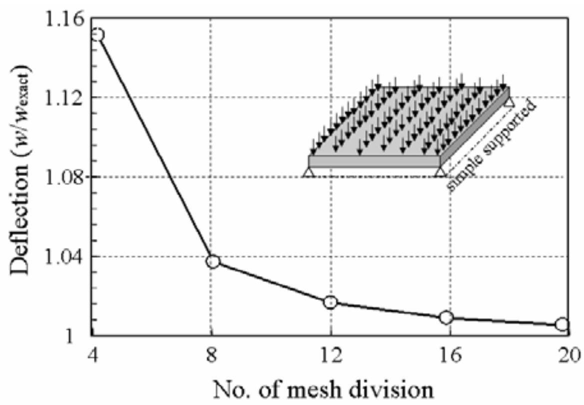

*Fig. 13 Ratio of deflections by exact and HPM when Poisson Ratio=0.1*

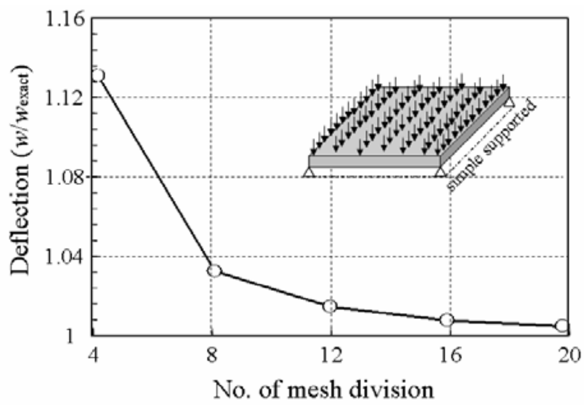

*Fig. 14 Ratio of deflections by exact and HPM when Poisson Ratio=0.2*

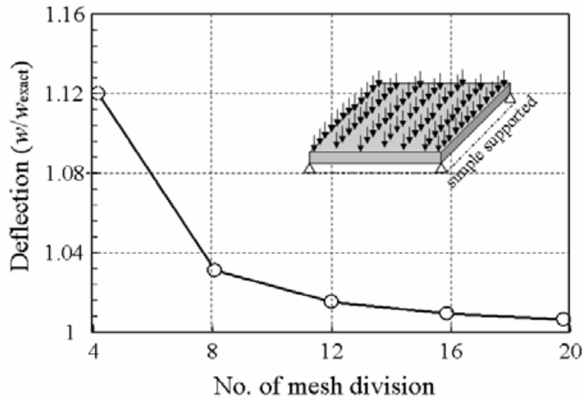


Fig. 15 Ratio of deflections by exact and HPM
when Poisson Ratio=0.3

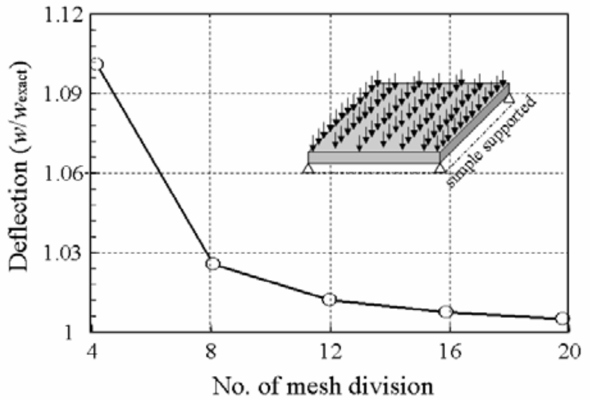


Fig. 16 Ratio of deflections by exact and HPM
when Poisson Ratio=0.4

Finally, it gives dependence between the deflection and Poisson Ratio:

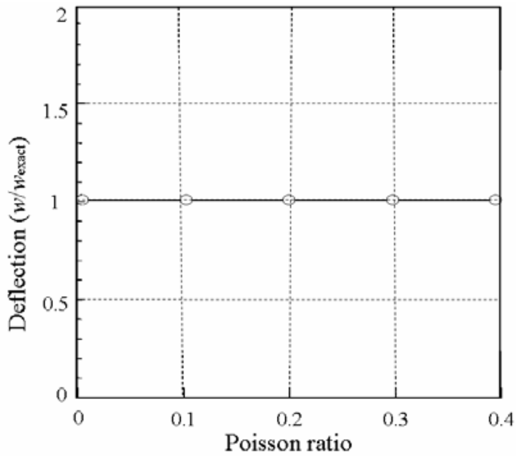


*Fig. 17 Deflection Dependence from Poisson Ratio*

## 5. CONCLUSIONS

In this paper proposed new approach for solving plate bending problem by using HPM. After comparison of analytical solution and HPM results for deflection and bending moment in case of several examples, we can see that we have high accuracy between them. As a result, we can conclude that HPM corresponds to all requirements for solving problems such as plate bending.

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